

# On completely faithful Selmer groups of elliptic curves and Hida deformations

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## Abstract

In this paper, we study completely faithful torsion  $\mathbb{Z}_p[[G]]$ -modules with applications to the study of Selmer groups. Namely, if  $G$  is a nonabelian group belonging to certain classes of polycyclic pro- $p$  group, we establish the abundance of faithful torsion  $\mathbb{Z}_p[[G]]$ -modules, i.e., non-trivial torsion modules whose global annihilator ideal is zero. We then show that such  $\mathbb{Z}_p[[G]]$ -modules occur naturally in arithmetic, namely in the form of Selmer groups of elliptic curves and Selmer groups of Hida deformations. It is interesting to note that faithful Selmer groups of Hida deformations do not seem to appear in literature before. We will also show that faithful Selmer groups have various arithmetic properties. Namely, we show that faithfulness is an isogeny invariant, and we will prove “control theorem” results on the faithfulness of Selmer groups over a general strongly admissible  $p$ -adic Lie extension.

Keywords and Phrases: Completely faithful modules, Selmer groups, elliptic curves, Hida deformations.

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## 1 Introduction

Throughout the paper,  $p$  will always denote an odd prime. Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$  which has good ordinary reduction at the prime  $p$ . The Iwasawa main conjecture predicts that the Mazur-Swinnerton-Dyer  $p$ -adic  $L$ -function  $\mathcal{L}_p(E)$  associated to  $E$  can be interpreted as an element of the Iwasawa-algebra  $\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})]]$  of the cyclotomic  $\mathbb{Z}_p$ -extension  $\mathbb{Q}^{\text{cyc}}$  of  $\mathbb{Q}$  and is a generator of the characteristic ideal of the Pontryagin dual  $X(E/\mathbb{Q}^{\text{cyc}})$  of the Selmer group of  $E$  over  $\mathbb{Q}^{\text{cyc}}$  (see [27]). Furthermore, if  $X(E/\mathbb{Q}^{\text{cyc}})$  does not have any nonzero pseudo-null submodule, it will follow from the main conjecture that the  $p$ -adic  $L$ -function  $\mathcal{L}_p(E)$  annihilates  $X(E/\mathbb{Q}^{\text{cyc}})$ . We should mention that the Iwasawa main conjecture in this context has been well understood and largely proven (see [20, 29, 31]).

It is natural to consider generalization of the above by considering field extensions  $F_\infty$  of some number field  $F$  whose Galois group  $G = \text{Gal}(F_\infty/F)$  is a nonabelian  $p$ -adic Lie group, and this has been the central theme in noncommutative Iwasawa theory. One of the earliest approach towards understanding and formulating this theory is to investigate the global annihilator ideal of  $X(E/F_\infty)$  (for instance, see

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[5, 15]). Inspired by the cyclotomic situation, it was then hoped that such an investigation might give some insight to the noncommutative  $p$ -adic  $L$ -function which is, even today, still largely conjectural in most situations (although one now has a slightly better understanding of the shape of the  $p$ -adic  $L$ -functions and the form of the main conjecture via an algebraic  $K$ -theoretical approach; see [3, 4, 8, 21]). As it turns out, such an approach via global annihilators had been shown to be *not* feasible in general. In fact, Venjakob was able to establish the existence of a class of modules over the Iwasawa algebra of the nonabelian group  $\mathbb{Z}_p \rtimes \mathbb{Z}_p$  which *cannot* be annihilated by a single global element in the Iwasawa algebra (see [33]). Building on this work, he and Hachimori were able to give examples of dual Selmer groups of elliptic curves over a false Tate extension which do not have a nonzero global annihilator (see [14]).

In this paper, following the footsteps of Venjakob, we will establish the nonexistence of global annihilators for a class of modules over the Iwasawa algebra of a nonabelian group which is an extension of a polycyclic pro- $p$  group by  $\mathbb{Z}_p \rtimes \mathbb{Z}_p$  (see Theorem 3.3). The technique we used in establishing this result derives essentially from [33]. In fact, to prove our main result, we will also require the case of  $\mathbb{Z}_p \rtimes \mathbb{Z}_p$  which was first established by Venjakob in [33]. Then as in the paper of Hachimori and Venjakob [14], we apply our result to obtain examples of completely faithful Selmer groups of elliptic curves and Hida deformations over noncommutative  $p$ -adic Lie extensions (see Theorems 4.1 and 4.4). To the best of the author's knowledge, completely faithful Selmer groups of Hida deformations do not seem to be observed in literature before. We mention that we can also find examples of Selmer groups of elliptic curves which are faithful but not completely faithful. We also mention that our results can be applied to obtain completely faithful Selmer groups for  $p$ -adic representations defined over coefficient rings  $\mathbb{Z}_p[[X_1, X_2, \dots, X_n]]$ . However, in this paper, we will content ourselves mainly with Hida deformations and a short remark in the general aspect.

For the remainder of the paper, we discuss further properties on the faithfulness of Selmer groups. In particular, we show that faithfulness is an isogeny invariant (see Proposition 5.1). On the other hand, we will give an example to show that completely faithfulness is not an isogeny invariant. In the final section of the paper, we will prove some “control theorem” type results for the faithfulness of Selmer groups (see Propositions 6.3 and 6.7). It would seem that “control theorem” type results for faithfulness of Selmer group have not been observed in literature before.

We should also mention that completely faithful modules and Selmer groups of elliptic curves over Iwasawa algebras of compact  $p$ -adic Lie group other than the ones considered in this paper have also been studied in [1, 2]. Our results here may therefore be viewed as complement to the results there.

We end the introductory section discussing some (negative) consequences and significance of our results. Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$  which has good ordinary reduction at the prime  $p$  and set  $F = \mathbb{Q}(\mu_p)$ . Let  $F_\infty$  be a false Tate extension of  $F$  in the sense of [14]. As shown loc. cit., there are cases of  $X(E/F_\infty)$  being completely faithful. One may then naively consider adjoining multiple  $\mathbb{Z}_p^r$ -extensions of  $F$  to  $F_\infty$  and perhaps hope to obtain nontrivial global annihilator of the Selmer groups which is now defined over a larger  $p$ -adic Lie extension. The rationale (which now seems irrational)

behind this thought is that our Selmer group is now a module over an Iwasawa algebra of the group  $\mathbb{Z}_p^r \times (\mathbb{Z}_p \rtimes \mathbb{Z}_p)$  which has a large “commutative” component and, therefore, one might naively hope that having large “commutative” component may somehow force the existence of nontrivial global annihilator for our Selmer group. However, as our results (both algebraic and arithmetic) will show, such an idea is not feasible in general.

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## 2 Algebraic Preliminaries

In this section, we establish some algebraic preliminaries and notation. Throughout the paper, we will always work with left modules over a ring. Let  $\Lambda$  be a (not necessarily commutative) Noetherian ring which has no zero divisors. Then it admits a skew field of fractions  $K(\Lambda)$  which is flat over  $\Lambda$  (see [9, Chapters 6 and 10] or [23, Chapter 4, §9 and §10]). If  $M$  is a finitely generated  $\Lambda$ -module, we define the  $\Lambda$ -rank of  $M$  to be

$$\text{rank}_\Lambda M = \dim_{K(\Lambda)} K(\Lambda) \otimes_\Lambda M.$$

Clearly, one has  $\text{rank}_\Lambda M = 0$  if and only if  $K(\Lambda) \otimes_\Lambda M = 0$ . We say that  $M$  is a *torsion*-module if  $\text{rank}_\Lambda M = 0$ . We shall record a simple lemma which is a special case of [24, Lemma 4.1].

**Lemma 2.1.** *Let  $x$  be a central element of  $\Lambda$  with the property that  $\Omega := \Lambda/x\Lambda$  also has no zero divisors. Let  $M$  be a finitely generated  $\Lambda$ -module. Then*

$$\text{rank}_\Omega M/xM = \text{rank}_\Omega M[x] + \text{rank}_\Lambda M,$$

where  $M[x]$  is the submodule of  $M$  killed by  $x$ .

For a nonzero  $\Lambda$ -module  $M$ , we define the global annihilator ideal

$$\text{Ann}_\Lambda(M) = \{\lambda \in \Lambda : \lambda m = 0 \text{ for all } m \in M\}.$$

Note that this is a two-sided ideal of  $\Lambda$ . Indeed, for a given  $\lambda \in \text{Ann}_\Lambda(M)$ , we have  $\mu\lambda m = 0$  for every  $\mu \in \Lambda$  and  $m \in M$ . On the other hand, since  $\mu m \in M$ , we also have  $\lambda\mu m = 0$ . We will say that  $M$  is a *faithful*  $\Lambda$ -module if  $\text{Ann}_\Lambda(M) = 0$ .

Now if  $x \in \Lambda$ , we denote  $M[x]$  to be the set consisting of elements of  $M$  annihilated by  $x$ . If  $x$  is not central,  $M[x]$  is at most an additive subgroup of  $M$ . However, if we assume further that  $x\Lambda = \Lambda x$ , then it is easy to see that  $M[x]$  is a  $\Lambda$ -submodule of  $M$ . Indeed, given  $\lambda \in \Lambda$  and  $m \in M$ , it follows from the

hypothesis  $x\Lambda = \Lambda x$  that  $\lambda'x = x\lambda$  for some  $\lambda' \in \Lambda$ , and therefore, one has  $x\lambda m = \lambda'xm = 0$ . Continuing to assume that  $x\Lambda = \Lambda x$ , one can also verify that

$$xM = \{xm : m \in M\}$$

is a  $\Lambda$ -submodule of  $M$ , and that  $M/xM$  is a  $\Lambda/x\Lambda$ -module. We finally point out that  $x\Lambda$  is a two-sided ideal under the condition that  $x\Lambda = \Lambda x$ .

The following lemma is a natural generalization of [33, Lemma 4.5] and will be a crucial ingredient in proving our main results in Sections 3 and 6. It will be of interest to have an analogous statement for completely faithful modules (see Remark 6.4) but we are not able to establish such a statement at this point of writing.

**Lemma 2.2.** *Let  $x$  be a nonzero element of  $\Lambda$  with the property that  $x\Lambda = \Lambda x$  and suppose that the ring  $\Omega := \Lambda/x\Lambda$  has no zero divisors. Write  $I = x\Lambda (= \Lambda x)$ . Let  $M$  be a finitely generated  $\Lambda$ -module. Suppose that  $M[x] = 0$ , and suppose that  $\cap_{i \geq 1} I^i = 0$ .*

*If  $M/xM$  is a faithful  $\Omega$ -module, then  $M$  is a faithful  $\Lambda$ -module.*

*Proof.* We will prove the contrapositive statement. Suppose that  $\text{Ann}_\Lambda(M)$  contains a nonzero element  $\lambda$ . Since  $\cap_{i \geq 1} I^i = 0$ , we can find  $n$  such that  $\lambda \in I^n$  but  $\lambda \notin I^{n+1}$ . This in turn implies that  $\lambda = x^n \lambda_0$  for some  $\lambda_0 \notin I$  (note that such a representation is possible by the assumption that  $x\Lambda = \Lambda x$ ). But since  $M[x] = 0$ , we actually have  $\lambda_0 \in \text{Ann}_\Lambda(M)$ . Since  $\lambda_0 \notin I$ , the image of  $\lambda_0$  in  $\Omega$  is nonzero and lies in  $\text{Ann}_\Omega(M/xM)$ .  $\square$

We record two more lemmas. The first has an easy proof which is left to the reader.

**Lemma 2.3.** *Suppose that we are given an exact sequence*

$$0 \longrightarrow B' \longrightarrow M' \longrightarrow M \longrightarrow B \longrightarrow 0$$

*of  $\Lambda$ -modules, where  $B'$  and  $B$  are both annihilated by a nonzero central element  $\lambda$  of  $\Lambda$ . Then  $M'$  is faithful over  $\Lambda$  if and only if  $M$  is faithful over  $\Lambda$ .*

**Lemma 2.4.** *Suppose that we are given an exact sequence*

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*of  $\Lambda$ -modules, where  $M'$  is nonzero and  $M''$  is finite. Then if  $M$  is faithful over  $\Lambda$ , so is  $M'$ .*

*Proof.* If the ring  $\Lambda$  has characteristic zero, one can probably give a proof along the line of the proof of Lemma 2.3. In fact, for much of the discussion in the paper, this case will suffice. However, we thought that it may be of interest to give a proof that works in general which we do now. We will prove the contrapositive statement. Suppose that  $\text{Ann}_\Lambda(M')$  contains a nonzero element  $\lambda$ . Since  $M''$  is finite, it follows that for each  $z \in M''$ , there exists  $n_z < m_z$  such that  $\lambda^{n_z} z = \lambda^{m_z} z$ . This in turn implies that  $\lambda^{n_z} (\lambda^{m_z - n_z} - 1) z = 0$ . Set  $n = 1 + \max_{z \in M'' \setminus \{0\}} n_z$  and  $m = \prod_{z \in M'' \setminus \{0\}} (m_z - n_z)$ . Clearly, one has

$\lambda^n(\lambda^m - 1)z = 0$  for every  $z \in M''$ . Also, since  $n > 0$  by our choice, we have that  $\lambda^n(\lambda^m - 1)$  lies in  $\text{Ann}_\Lambda(M')$ . Therefore,  $\lambda^n(\lambda^m - 1)$  lies in  $\text{Ann}_\Lambda(M)$ . It remains to show that  $\lambda^n(\lambda^m - 1)$  is a nonzero element of  $\Lambda$ . Let  $w$  be a nonzero element of  $M'$  (such an element exists by our hypothesis that  $M' \neq 0$ ). Then  $(\lambda^m - 1)w = -w \neq 0$ , and this in turn implies that  $\lambda^m - 1 \neq 0$ . Since  $\Lambda$  has no zero divisors and  $\lambda \neq 0$ , it follows that  $\lambda^n(\lambda^m - 1)$  is also nonzero. This completes the proof of the lemma.  $\square$

Let  $\Lambda$  be a Auslander regular ring (see [32, Definition 3.3]) with no zero divisors. Let  $M$  be a finitely generated  $\Lambda$ -module. Then  $M$  is a torsion  $\Lambda$ -module if and only if  $\text{Hom}_\Lambda(M, \Lambda) = 0$  (cf. [24, Lemma 4.2]). If  $M$  is a torsion  $\Lambda$ -module, we say that  $M$  is a *pseudo-null*  $\Lambda$ -module if  $\text{Ext}_\Lambda^1(M, \Lambda) = 0$ . Let  $\mathcal{M}$  denote the category of all finitely generated  $\Lambda$ -modules, let  $\mathcal{C}$  denote the full subcategory of all pseudo-null modules in  $\mathcal{M}$  and let  $q : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$  denote the quotient functor. For a finitely generated  $\Lambda$ -module  $M$ , we say that  $M$  is *completely faithful* if  $\text{Ann}_\Lambda(N) = 0$  for any  $N \in \mathcal{M}$  such that  $q(N)$  is isomorphic to a non-zero subquotient of  $q(M)$ .

**Lemma 2.5.** *Let  $\Lambda$  be an Auslander regular ring with no zero divisors. Then we have the following statements.*

- (a) *If  $M$  is completely faithful over  $\Lambda$ , so is every non pseudo-null subquotient of  $M$ .*
- (b) *An extension of completely faithful  $\Lambda$ -modules is also completely faithful.*

*Proof.* This is straightforward from the definition.  $\square$

### 3 Completely faithful modules over completed group algebras

In this section, we will prove our main theorem which is an extension of [33, Theorem 6.3]. As before,  $p$  will denote a fixed odd prime. Let  $G$  be a compact pro- $p$   $p$ -adic Lie group without  $p$ -torsion. It is well known that  $\mathbb{Z}_p[[G]]$  is an Auslander regular ring (cf. [32, Theorems 3.26]). Furthermore, the ring  $\mathbb{Z}_p[[G]]$  has no zero divisors (cf. [28]), and therefore, as seen in the previous section, there is a well-defined notion of  $\mathbb{Z}_p[[G]]$ -rank and torsion  $\mathbb{Z}_p[[G]]$ -module. We record the following well-known and important result of Venjakob (cf. [33, Example 2.3 and Proposition 5.4]).

**Theorem 3.1** (Venjakob). *Suppose that  $H$  is a closed normal subgroup of  $G$  with  $G/H \cong \mathbb{Z}_p$ . Let  $M$  be a compact  $\mathbb{Z}_p[[G]]$ -module which is finitely generated over  $\mathbb{Z}_p[[H]]$ . Then  $M$  is a pseudo-null  $\mathbb{Z}_p[[G]]$ -module if and only if  $M$  is a torsion  $\mathbb{Z}_p[[H]]$ -module.*

We record another useful lemma whose proof is left to the reader (or see [24, Lemma 4.5]).

**Lemma 3.2.** *Let  $H$  be a compact pro- $p$   $p$ -adic Lie group without  $p$ -torsion. Let  $N$  be a closed normal subgroup of  $H$  such that  $N \cong \mathbb{Z}_p$  and such that  $H/N$  is also a compact pro- $p$   $p$ -adic Lie group without  $p$ -torsion. Let  $M$  be a finitely generated  $\mathbb{Z}_p[[H]]$ -module. Then  $H_i(N, M)$  is finitely generated over  $\mathbb{Z}_p[[H/N]]$  for each  $i$  and  $H_i(N, M) = 0$  for  $i \geq 2$ . Furthermore, we have an equality*

$$\text{rank}_{\mathbb{Z}_p[[H]]} M = \text{rank}_{\mathbb{Z}_p[[H/N]]} M_N - \text{rank}_{\mathbb{Z}_p[[H/N]]} H_1(N, M).$$

Before continuing our discussion, we introduce the following hypothesis on our group  $G$ .

(NH) : The group  $G$  contains two closed normal subgroups  $N$  and  $H$  which satisfy the following two properties.

- (i)  $N \subseteq H$ ,  $G/H \cong \mathbb{Z}_p$  and  $G/N$  is a non-abelian group isomorphic to  $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ .
- (ii) There is a finite family of closed normal subgroups  $N_i$  ( $0 \leq i \leq r$ ) of  $G$  such that  $1 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_r = N$  and such that  $N_i/N_{i-1} \cong \mathbb{Z}_p$  for  $1 \leq i \leq r$ .

We can now state and prove the main theorem of this section which generalizes [33, Theorem 6.3].

**Theorem 3.3.** *Suppose that  $G$  satisfies (NH). Let  $M$  be a  $\mathbb{Z}_p[[G]]$ -module which is finitely generated over  $\mathbb{Z}_p[[H]]$  and has positive  $\mathbb{Z}_p[[H]]$ -rank. Then  $M$  is a completely faithful  $\mathbb{Z}_p[[G]]$ -module.*

*Proof.* Since  $M$  is finitely generated over  $\mathbb{Z}_p[[H]]$ , it follows from the result of Venjakob mentioned above that every subquotient of  $M$  which is not pseudo-null has positive  $\mathbb{Z}_p[[H]]$ -rank. Therefore, it suffices to show that every  $\mathbb{Z}_p[[G]]$ -module that is finitely generated over  $\mathbb{Z}_p[[H]]$  with positive  $\mathbb{Z}_p[[H]]$ -rank is faithful over  $\mathbb{Z}_p[[G]]$ . We will proceed by induction on  $r$ . When  $r = 0$ , this is precisely [33, Corollary 4.3]. Now suppose that  $r \geq 1$  and suppose that  $M$  is finitely generated over  $\mathbb{Z}_p[[H]]$  with positive  $\mathbb{Z}_p[[H]]$ -rank. Then choose a topological generator  $\gamma_1$  of  $N_1$ . The ideal generated by  $\gamma_1 - 1$  is precisely the augmentation kernel  $I_{N_1}$  of the canonical quotient map  $\mathbb{Z}_p[[G]] \twoheadrightarrow \mathbb{Z}_p[[G/N_1]]$ , and one has  $I_{N_1} = (\gamma_1 - 1)\mathbb{Z}_p[[G]] = \mathbb{Z}_p[[G]](\gamma_1 - 1)$ . Therefore,  $M[\gamma_1 - 1]$  is a  $\mathbb{Z}_p[[G]]$ -submodule of  $M$ . Suppose for now that  $M[\gamma_1 - 1] = 0$ . Then it follows from Lemma 3.2 that

$$\text{rank}_{\mathbb{Z}_p[[H/N_1]]} M_{N_1} = \text{rank}_{\mathbb{Z}_p[[G]]} M + \text{rank}_{\mathbb{Z}_p[[H/N_1]]} H_1(N_1, M) = \text{rank}_{\mathbb{Z}_p[[H]]} M > 0.$$

Here the second equality follows from the facts that  $H_1(N_1, M) = M[\gamma_1 - 1]$  and that  $M[\gamma_1 - 1] = 0$ . Hence  $M_{N_1}$  is a  $\mathbb{Z}_p[[G/N_1]]$ -module which is finitely generated over  $\mathbb{Z}_p[[H/N_1]]$  with positive  $\mathbb{Z}_p[[H/N_1]]$ -rank. By our induction hypothesis, we have that  $M_{N_1}$  is (completely) faithful over  $\mathbb{Z}_p[[G/N_1]]$ . Note that  $I_{N_1}$  is closed in  $\mathbb{Z}_p[[G]]$  and so  $\cap_{i \geq 1} I_{N_1}^i = 0$ . Also, since we are assuming that  $M[\gamma_1 - 1] = 0$ , we may apply Lemma 2.2 to conclude that  $M$  is faithful over  $\mathbb{Z}_p[[G]]$ .

It remains to consider the situation when  $M[\gamma_1 - 1] \neq 0$ . Since  $(\gamma_1 - 1)^i \mathbb{Z}_p[[G]] = \mathbb{Z}_p[[G]](\gamma_1 - 1)^i$  for all  $i \geq 1$ , it follows that  $M[(\gamma_1 - 1)^i]$  is a  $\mathbb{Z}_p[[G]]$ -submodule of  $M$  for every  $i \geq 1$ . As  $M$  is a Noetherian  $\mathbb{Z}_p[[G]]$ -module, the chain

$$M[\gamma_1 - 1] \subseteq M[(\gamma_1 - 1)^2] \subseteq \dots \subseteq M[(\gamma_1 - 1)^i] \subseteq \dots$$

terminates at a finite level which in turn implies that

$$M' := \cup_{i \geq 1} M[(\gamma_1 - 1)^i] = M[(\gamma_1 - 1)^n]$$

for some  $n$ . Set  $M'' = M/M'$ . Since  $M' = M[(\gamma_1 - 1)^n]$  is a torsion  $\mathbb{Z}_p[[H]]$ -module, it follows that  $M''$  has positive  $\mathbb{Z}_p[[H]]$ -rank. On the other hand, one clearly has  $\text{Ann}_{\mathbb{Z}_p[[G]]}(M) \subseteq \text{Ann}_{\mathbb{Z}_p[[G]]}(M'')$ , and

therefore, we are reduced to showing that  $M''$  is faithful over  $\mathbb{Z}_p[[G]]$ . By our construction of  $M''$ , we have that  $M''[\gamma_1 - 1] = 0$ . Hence we may apply the argument in the previous paragraph to obtain the faithfulness of  $M''$ . The proof of the theorem is now complete.  $\square$

**Remark 3.4.** One can of course prove analogous result as in Theorem 3.3 replacing the coefficient ring  $\mathbb{Z}_p$  by  $\mathbb{F}_p$ . This is achieved by combining the argument in Theorem 3.3 with [33, Proposition 4.2(i)]. In fact, it is not difficult (though not immediate) to see that one can also prove analogous results as in the paper [33] and Theorem 3.3 for a ring of integer of a finite extension of  $\mathbb{Q}_p$  and its residue field.

We should mention here that the conclusion in Theorem 3.3 does not hold for a general  $G$ . We give a class of counterexamples. For the remainder of this paragraph, we assume that  $G = H \times G/H$ . Let  $M = \mathbb{Z}_p[[H]]$  be the  $\mathbb{Z}_p[[G]]$ -module where the action of  $H$  is the natural one, and the action of  $G/H$  is the trivial one. Denote  $\gamma$  to be a topological generator of  $G/H$ . Clearly,  $M$  is clearly annihilated by  $\gamma - 1$ , and so it is not faithful over  $\mathbb{Z}_p[[G]]$ .

Despite the counterexamples, it is still of interest to ask if there exists other torsionfree pro- $p$   $p$ -adic Lie group  $G$  where a similar conclusion in Theorem 3.3 holds. Alternatively, one may ask if  $G$  is a torsionfree pro- $p$   $p$ -adic Lie group with a normal subgroup  $H$  such that  $G/H \cong \mathbb{Z}_p$ , and has the property that every finitely generated  $\mathbb{Z}_p[[H]]$ -module of positive  $\mathbb{Z}_p[[H]]$ -rank is completely faithful over  $\mathbb{Z}_p[[G]]$ , what can one say about the structure of  $G$ ? The author does not have an answer to these questions at this point of writing.

Finally, we end the section mentioning how the results in this section can be extended to a certain class of finitely generated torsion  $\mathbb{Z}_p[[G]]$  which was first introduced in [4]. In particular, this class of modules is a source of examples of *faithful modules that are not completely faithful*. As before,  $G$  is a compact pro- $p$   $p$ -adic Lie group without  $p$ -torsion and  $H$  is a closed normal subgroup of  $G$  such that  $G/H \cong \mathbb{Z}_p$ . For a finitely generated  $\mathbb{Z}_p[[G]]$ -module  $M$ , we say that  $M$  belongs to  $\mathfrak{M}_H(G)$  if  $M/M(p)$  is finitely generated over  $\mathbb{Z}_p[[H]]$ . Here  $M(p)$  is the submodule of  $M$  consisting of elements of  $M$  annihilated by a power of  $p$ . Note that a finitely generated  $\mathbb{Z}_p[[G]]$ -module belonging to  $\mathfrak{M}_H(G)$  is necessarily a torsion  $\mathbb{Z}_p[[G]]$ -module.

**Corollary 3.5.** *Suppose that  $G$  satisfies (NH). Let  $M$  be a  $\mathbb{Z}_p[[G]]$ -module which belongs to  $\mathfrak{M}_H(G)$  and has the property that  $M/M(p)$  has a positive  $\mathbb{Z}_p[[H]]$ -rank. Then  $M$  is a faithful  $\mathbb{Z}_p[[G]]$ -module.*

*Furthermore,  $M$  is a completely faithful  $\mathbb{Z}_p[[G]]$ -module if and only if  $M(p)$  is a pseudo-null  $\mathbb{Z}_p[[G]]$ -module.*

*Proof.* Since  $M$  is finitely generated over  $\mathbb{Z}_p[[G]]$ , the module  $M(p)$  is annihilated by a power of  $p$ . The first assertion is now an immediate consequence from Lemma 2.3 and Theorem 3.3.

To prove the second assertion, we first recall that  $q : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$  is the quotient functor, where  $\mathcal{M}$  denotes the category of all finitely generated  $\Lambda$ -modules and  $\mathcal{C}$  denotes the full subcategory of all pseudo-null modules in  $\mathcal{M}$ . Now suppose that  $M(p)$  is a pseudo-null  $\mathbb{Z}_p[[G]]$ -module, then we have  $q(M) = q(M/M(p))$ . Since  $M/M(p)$  is completely faithful by Theorem 3.3, it follows that  $M$  is also completely faithful. On the other hand, if  $M(p)$  is not a pseudo-null  $\mathbb{Z}_p[[G]]$ -module and is annihilated by

some power of  $p$ , then  $M$  contains a submodule which is not pseudo-null and not faithful, and therefore, is not completely faithful.  $\square$

**Remark 3.6.** Note that  $M(p)$  is a pseudo-null  $\mathbb{Z}_p[[G]]$ -module if and only if its  $\mu_G$ -invariant (see [32, Definition 3.32] for definition) vanishes (cf. [32, Remark 3.33]).

## 4 Completely faithful Selmer groups

Let  $F$  be a number field. Fix once and for all an algebraic closure  $\bar{F}$  of  $F$ . Therefore, an algebraic (possibly infinite) extension of  $F$  will mean an subfield of  $\bar{F}$  which contains  $F$ . Let  $E$  be an elliptic curve over  $F$ . Assume that for every prime  $v$  of  $F$  above  $p$ , our elliptic curve  $E$  has either good ordinary reduction or multiplicative reduction at  $v$ .

Let  $v$  be a prime of  $F$ . For every finite extension  $L$  of  $F$ , we define

$$J_v(E/L) = \bigoplus_{w|v} H^1(L_w, E)_{p^\infty},$$

where  $w$  runs over the (finite) set of primes of  $L$  above  $v$ . If  $\mathcal{L}$  is an infinite extension of  $F$ , we define

$$J_v(E/\mathcal{L}) = \varinjlim_L J_v(E/L),$$

where the direct limit is taken over all finite extensions  $L$  of  $F$  contained in  $\mathcal{L}$ . For any algebraic (possibly infinite) extension  $\mathcal{L}$  of  $F$ , the Selmer group of  $E$  over  $\mathcal{L}$  is defined to be

$$S(E/\mathcal{L}) = \ker \left( H^1(\mathcal{L}, E_{p^\infty}) \longrightarrow \bigoplus_v J_v(E/\mathcal{L}) \right),$$

where  $v$  runs through all the primes of  $F$ .

We say that  $F_\infty$  is an *admissible  $p$ -adic Lie extension* of  $F$  if (i)  $\text{Gal}(F_\infty/F)$  is a compact  $p$ -adic Lie group, (ii)  $F_\infty$  contains the cyclotomic  $\mathbb{Z}_p$ -extension  $F^{\text{cyc}}$  of  $F$  and (iii)  $F_\infty$  is unramified outside a finite set of primes of  $F$ . Furthermore, an admissible  $p$ -adic Lie extension  $F_\infty$  of  $F$  will be said to be *strongly admissible* if  $\text{Gal}(F_\infty/F)$  is a compact pro- $p$   $p$ -adic Lie group without  $p$ -torsion. Write  $G = \text{Gal}(F_\infty/F)$ ,  $H = \text{Gal}(F_\infty/F^{\text{cyc}})$  and  $\Gamma = \text{Gal}(F^{\text{cyc}}/F)$ . Let  $S$  be a finite set of primes of  $F$  which contains the primes above  $p$ , the infinite primes, the primes at which  $E$  has bad reduction and the primes that are ramified in  $F_\infty/F$ . Denote  $F_S$  to be the maximal algebraic extension of  $F$  unramified outside  $S$ . For each algebraic (possibly infinite) extension  $\mathcal{L}$  of  $F$  contained in  $F_S$ , we write  $G_S(\mathcal{L}) = \text{Gal}(F_S/\mathcal{L})$ . The following alternative equivalent description of the Selmer group of  $E$  over  $F_\infty$

$$S(E/F_\infty) = \ker \left( H^1(G_S(F_\infty), E_{p^\infty}) \xrightarrow{\lambda_{E/F_\infty}} \bigoplus_{v \in S} J_v(E/F_\infty) \right)$$

is well-known (for instance, see [7, Lemma 2.2]). We will denote  $X(E/F_\infty)$  to be the Pontryagin dual of  $S(E/F_\infty)$ . The following is then an immediate consequence of Theorem 3.3.



**Theorem 4.1.** *Let  $E$  be an elliptic curve over  $F$  which has either good ordinary reduction or multiplicative reduction at every prime of  $F$  above  $p$ . Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with  $G = \text{Gal}(F_\infty/F)$ . Suppose that  $G$  satisfies (NH). If  $X(E/F_\infty)$  is finitely generated over  $\mathbb{Z}_p[[H]]$  with positive  $\mathbb{Z}_p[[H]]$ -rank, then  $X(E/F_\infty)$  is completely faithful over  $\mathbb{Z}_p[[G]]$ .*

We record the following corollary of the theorem which is useful in obtaining examples of completely faithful Selmer groups.

**Corollary 4.2.** *Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with  $G = \text{Gal}(F_\infty/F)$ . Suppose that  $G$  satisfies (NH). Assume that  $X(E/F^{\text{cyc}})$  is finitely generated over  $\mathbb{Z}_p$ . Furthermore, suppose that either of the following conditions is satisfied.*

- (a)  *$X(E/F^{\text{cyc}})$  has positive  $\mathbb{Z}_p$ -rank.*
- (b) *The field  $F$  is not totally real, the elliptic curve  $E$  has good ordinary reduction at every prime of  $F$  above  $p$  and  $X(E/F_\infty) \neq 0$ .*

*Then  $X(E/F_\infty)$  is completely faithful over  $\mathbb{Z}_p[[G]]$ .*

*Proof.* By [4, Proposition 5.6] or [7, Theorem 2.1], if  $X(E/F^{\text{cyc}})$  is finitely generated over  $\mathbb{Z}_p$ , then  $X(E/F_\infty)$  is finitely generated over  $\mathbb{Z}_p[[H]]$ . It suffices to show that  $X(E/F_\infty)$  has positive  $\mathbb{Z}_p[[H]]$ -rank under the assumption of either conditions. If (a) holds, a standard argument in the spirit of [11] will allow us establish the positivity of  $\mathbb{Z}_p[[H]]$ -rank (alternatively, one can make use of [13, Theorem 5.4] directly). Now if (b) holds, we may apply the main result in [26] to conclude that  $X(E/F_\infty)$  has positive  $\mathbb{Z}_p[[H]]$ -rank (for instance, see [25, Lemma 5.8]).  $\square$

**Remark 4.3.** Of course, one can apply Theorem 3.3 to obtain completely faithful Selmer groups of  $p$ -ordinary modular forms, or even more general  $p$ -adic representations.

We now discuss the complete faithfulness of Selmer groups of Hida deformations. Let  $E$  be an elliptic curve over  $\mathbb{Q}$  with ordinary reduction at  $p$  and assume that  $E[p]$  is an absolutely irreducible  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -representation. By Hida theory (for instance, see [16, 17]), there exists a commutative complete Noetherian local domain  $R$  which is flat over the power series ring  $\mathbb{Z}_p[[X]]$  in one variable, and a free  $R$ -module  $T$  of rank 2 with  $T/P \cong T_p E$  for some prime ideal  $P$  of  $R$ . *We will further assume that  $R = \mathbb{Z}_p[[X]]$  in all our discussion.* For more detailed description of fundamental and important arithmetic properties of the Hida deformations, we refer readers to [7, 16, 17, 19, 30]. We will just mention two properties of  $T$  which we require to define an appropriate Selmer group of the Hida deformation. The first is that  $T$  is unramified outside the set  $S$ , where  $S$  is any finite set of primes of  $F$  which contains the primes above  $p$ , the infinite primes, the primes at which  $E$  has bad reduction and the primes that are ramified in  $F_\infty/F$ . The second property we will mention is that there exists an  $R$ -submodule  $T^+$  of  $T$  which is invariant under the action of  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  and such that both  $T^+$  and  $T/T^+$  are free  $R$ -modules of rank one.

Set  $A = T \otimes_R \text{Hom}_{\text{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$  and  $A^+ = T^+ \otimes_R \text{Hom}_{\text{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ . We note that one has  $E_{p^\infty} = A[P]$ . Then following [7, Section 4] or [30, Section 6], we define the Selmer group of the Hida deformation over an admissible  $p$ -adic Lie extension  $F_\infty$  of  $\mathbb{Q}$  by

$$S(A/F_\infty) = \ker \left( H^1(G_S(F_\infty), A) \longrightarrow \bigoplus_{v \in S} J_v(A, F_\infty) \right),$$

where

$$J_v(A, F_\infty) = \begin{cases} \prod_{w|v} H^1(F_{\infty,w}, A/A^+), & \text{if } v \text{ divides } p, \\ \prod_{w|v} H^1(F_{\infty,w}, A), & \text{if } v \text{ does not divide } p. \end{cases}$$

We will denote by  $X(A/F_\infty)$  the Pontryagin dual of this Selmer group. We will consider this dual Selmer group as a (compact)  $\text{Gal}(F_\infty/F)$ -module for some finite extension  $F$  of  $\mathbb{Q}$  in  $F_\infty$ , where  $F_\infty$  is a strongly admissible  $p$ -adic Lie extension of  $F$ .

**Theorem 4.4.** *Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with Galois group  $G$ . Suppose that  $G$  satisfies (NH). If  $X(A/F_\infty)$  is finitely generated over  $R[[H]]$  with positive  $R[[H]]$ -rank, then  $X(A/F_\infty)$  is completely faithful over  $R[[G]]$ . In particular, if  $X(E/F_\infty)$  is finitely generated over  $\mathbb{Z}_p[[H]]$  with positive  $\mathbb{Z}_p[[H]]$ -rank, then  $X(A/F_\infty)$  is completely faithful over  $R[[G]]$ .*

*Proof.* By identifying  $R[[G]] \cong \mathbb{Z}_p[[\mathbb{Z}_p \times G]]$ , the first part of the theorem is immediate from Theorem 3.3. For the second part, it suffices to show that  $X(A/F_\infty)$  is finitely generated over  $R[[H]]$  with positive  $R[[H]]$ -rank. By [7, Theorem 4.2], the map

$$X(A/F_\infty)/P \longrightarrow X(E/F_\infty)$$

has cokernel which is finitely generated over  $\mathbb{Z}_p$ . Since  $X(E/F_\infty)$  has positive  $\mathbb{Z}_p[[H]]$ -rank, this in turn implies that  $X(A/F_\infty)/P$  has positive  $\mathbb{Z}_p[[H]]$ -rank. By an application of an argument in the spirit to that in [30, Theorem 7.4], one can show that  $S(A/F_\infty)/P = 0$ . Equivalently, this is the same as saying that  $X(A/F_\infty)[P] = 0$ . Therefore, we may apply Lemma 2.1 to conclude that  $R[[H]]$ -rank of  $X(A/F_\infty)$  is the same as the  $\mathbb{Z}_p[[H]]$ -rank of  $X(E/F_\infty)$  and, in particular, is positive as required.  $\square$

**Remark 4.5.** One can also obtain completely faithful Selmer groups of  $p$ -adic representations defined over coefficient rings  $\mathbb{Z}_p[[X_1, X_2, \dots, X_n]]$  over strongly admissible  $p$ -adic Lie extensions of the form considered in this section.

We now give some examples to illustrate the results in this section.

(a) Let  $E$  be the elliptic curve 11a2 of Cremona's table which is given by

$$y^2 + y = x^3 - x.$$

Take  $p = 5$ ,  $F = \mathbb{Q}(\mu_5)$  and  $L_\infty = \mathbb{Q}(\mu_{5^\infty}, 11^{5^{-\infty}})$ . By [14, Theorem 6.2],  $X(E/L_\infty)$  is a free  $\mathbb{Z}_5[[\text{Gal}(L_\infty/F^{\text{cyc}})]]$ -module of rank four. Let  $F_\infty$  be a strongly admissible 5-adic Lie extension of  $F$  that contains  $L_\infty$  and that the group  $N = \text{Gal}(F_\infty/L_\infty)$  satisfies the conditions in Theorem 4.1. A similar argument as in Corollary 4.2 shows that  $X(E/F_\infty)$  is a finitely generated  $\mathbb{Z}_5[[H]]$ -module of positive rank. Then we may apply Theorem 4.1 to conclude that  $X(E/F_\infty)$  is completely faithful over  $\mathbb{Z}_5[[G]]$ .

Now if  $E'$  is either 11a1 or 11a3, then it follows from Proposition 5.1 that  $X(E'/F_\infty)$  is faithful over  $\mathbb{Z}_5[[G]]$ . We claim that in either cases,  $X(E'/F_\infty)$  is not completely faithful. To see this, it suffices, by Proposition 3.5, to show that  $X(E'/F_\infty)(5)$  is not pseudo-null, or equivalently, the  $\mu_G$ -invariant of  $X(E'/F_\infty)$  is positive. By an application of [4, Lemma 5.6], we have that  $X(E'/F_\infty)$  belongs to  $\mathfrak{M}_H(G)$ . This in turn allows us to apply [25, Theorem 3.1] to conclude that  $\mu_G(X(E'/F_\infty)) = \mu_\Gamma(X(E'/F_{\text{cyc}}))$ . But this latter quantity is well-known to be nonzero (cf. [6, Theorem 5.28]) and hence our claim is established.

(b) The next example is taken from [19]. Let  $E$  be the elliptic curve 79a1 of Cremona's table which is given by

$$y^2 + xy + y = x^3 + x^2 - 2x.$$

Take  $p = 3$  and  $F = \mathbb{Q}(\mu_3)$ . As noted in [19],  $X(E/F^{\text{cyc}})$  is isomorphic to  $\mathbb{Z}_3$ . Let  $F_\infty$  be a strongly admissible 3-adic Lie extension of  $F$  that satisfies the conditions in Theorem 4.1. Write  $G = \text{Gal}(F_\infty/F)$ . By Corollary 4.2,  $X(E/F_\infty)$  is a completely faithful  $\mathbb{Z}_3[[G]]$ -module. Let  $A$  be the Galois module obtained from the Hida family associated to  $E$  as above. Therefore, one may apply Theorem 4.4 to conclude that  $X(A/F_\infty)$  is a completely faithful  $R[[G]]$ -module. Examples of strongly admissible 3-adic extensions of  $F$  that one can take are:

$$\mathbb{Q}(\mu_{3^\infty}, 2^{3^{-\infty}}), \quad \mathbb{Q}(\mu_{3^\infty}, 2^{3^{-\infty}}, 3^{3^{-\infty}}), \quad \mathbb{Q}(\mu_{3^\infty}, 3^{3^{-\infty}}, 5^{3^{-\infty}}), \quad \mathbb{Q}(\mu_{3^\infty}, 2^{3^{-\infty}}, 3^{3^{-\infty}}, 5^{3^{-\infty}}), \quad \dots \text{ etc.}$$

## 5 Isogeny invariance of faithful Selmer groups

In this short section, we will show that the property of faithfulness is an isogeny invariant. Namely, we prove the following statement.

**Proposition 5.1.** *Let  $E_1$  and  $E_2$  be two elliptic curves over  $F$  with either good ordinary reduction or multiplicative reduction at every prime of  $F$  above  $p$  which are isogenous to each other. Let  $F_\infty$  be a strongly admissible noncommutative  $p$ -adic Lie extension of  $F$  with  $G = \text{Gal}(F_\infty/F)$ . Assume that both  $X(E_1/F_\infty)$  and  $X(E_2/F_\infty)$  are torsion over  $\mathbb{Z}_p[[G]]$ , and that the localization maps  $\lambda_{E_1/F_\infty}$  and  $\lambda_{E_2/F_\infty}$  are surjective. Then  $X(E_1/F_\infty)$  is a faithful  $\mathbb{Z}_p[[G]]$ -module if and only if  $X(E_2/F_\infty)$  is a faithful  $\mathbb{Z}_p[[G]]$ -module.*

*Proof.* Let  $\varphi : E_1 \rightarrow E_2$  be an isogeny defined over  $F$ . By a standard argument to that in the proof of [14, Theorem 5.1] or [18, Theorem 3.1], we can show that  $\varphi$  induces a  $\mathbb{Z}_p[[G]]$ -homomorphism

$$X(E_2/F_\infty) \rightarrow X(E_1/F_\infty),$$

whose kernel and cokernel are killed by  $p^n$  for some large enough  $n$ . The required conclusion is now immediate from an application of Lemma 2.3.  $\square$

On the other hand, completely faithfulness is not an isogeny invariant. As seen in the previous section, the dual Selmer group of 11a2 is completely faithful but the the dual Selmer group of 11a1 and 11a3 are not.

## 6 Control theorems for faithfulness of Selmer groups

In this section, we will prove two control theorems on faithfulness of Selmer groups which can be applied to a  $p$ -adic Lie extension whose Galois group does not satisfy **(NH)**. In particular, our result will show that one cannot obtain nonfaithful Selmer groups from a faithful Selmer group by adjoining  $\mathbb{Z}_p^r$ -extension or moving into the Hida deformation in general. We retain the notation of the Section 4. Recall that  $S$  is a finite set of primes of  $F$  which contains the primes above  $p$ , the infinite primes, the primes at which  $E$  has bad reduction and the primes that ramify in  $F_\infty/F$ . We now record a lemma.

**Lemma 6.1.** *Let  $E$  be an elliptic curve over  $F$  with either good ordinary reduction or multiplicative reduction at every prime of  $F$  above  $p$ . Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with  $G = \text{Gal}(F_\infty/F)$ . Let  $N$  be a closed normal subgroup of  $G$  such that  $N \cong \mathbb{Z}_p$  and that  $G/N$  is a pro- $p$   $p$ -adic Lie group without  $p$ -torsion. Set  $L_\infty = F_\infty^N$ . Consider the following statements.*

- (i)  $X(E/L_\infty)$  is a torsion  $\mathbb{Z}_p[[G/N]]$ -module.
- (ii)  $E_{p^\infty}$  is not rational over  $L_\infty$ .
- (iii) For every  $v \in S$ , the decomposition group of  $G$  at  $v$  has dimension  $\geq 2$ .

If (i) holds, then  $X(E/F_\infty)$  is a torsion  $\mathbb{Z}_p[[G]]$ -module. If (i) and (ii) hold, or (i) and (iii) hold, then we have that  $H_1(N, X(E/F_\infty)) = 0$ .

*Proof.* By an argument similar to that in [7, Lemma 2.4], we have that the dual of the restriction map

$$X(E/F_\infty)_N \longrightarrow X(E/L_\infty)$$

has kernel and cokernel which are finitely generated over  $\mathbb{Z}_p[[H/N]]$ . Therefore, if (i) holds, it then follows that  $X(E/F_\infty)_N$  is a torsion  $\mathbb{Z}_p[[G/N]]$ -module. By Lemma 3.2, this in turns implies that  $X(E/F_\infty)$  is a torsion  $\mathbb{Z}_p[[G]]$ -module.

Now if we suppose that either (i) and (ii) hold, or (i) and (iii) hold. Then by [25, Proposition 3.3], we have a short exact sequence

$$0 \longrightarrow S(E/L_\infty) \longrightarrow H^1(G_S(L_\infty), E_{p^\infty}) \xrightarrow{\lambda_{E/L_\infty}} \bigoplus_{v \in S} J_v(E/L_\infty) \longrightarrow 0$$

and that  $H^2(G_S(L_\infty), E_{p^\infty}) = 0$ . By [24, Lemma 7.1, Proposition 7.2], the latter in turn implies that  $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ . Now we may apply a similar argument to that in [30, Theorem 7.4] (or [25, Lemma 4.4, Proposition 4.5]) to obtain the conclusion that  $H_1(N, X(E/F_\infty)) = 0$ .  $\square$

**Remark 6.2.** We can obtain  $H_1(N, X(E/F_\infty)) = 0$  without assumptions (ii) and (iii), provided one assumes that  $X(E/L_\infty)$  belongs to  $\mathfrak{M}_{H/N}(G/N)$ . To see this, we first note that since  $X(E/L_\infty)$  belongs to  $\mathfrak{M}_{H/N}(G/N)$ , it follows from [7, Proposition 2.5] that, for every finite extension  $K$  of  $F$  in  $L_\infty$ ,

$X(E/K^{\text{cyc}})$  is  $\mathbb{Z}_p[[\Gamma_K]]$ -torsion, where  $\Gamma_K = \text{Gal}(K^{\text{cyc}}/K)$ . By [25, Corollary 3.4], this in turn implies that we have a short exact sequence

$$0 \longrightarrow S(E/L_\infty) \longrightarrow H^1(G_S(L_\infty), E_{p^\infty}) \xrightarrow{\lambda_{E/L_\infty}} \bigoplus_{v \in S} J_v(E/L_\infty) \longrightarrow 0$$

and  $H^2(G_S(L_\infty), E_{p^\infty}) = 0$ . Now we may proceed as in the argument to that in Lemma 6.1 to obtain the conclusion that  $H_1(N, X(E/F_\infty)) = 0$ .

We can now prove our first control theorem which concerns extensions of admissible  $p$ -adic Lie extensions.

**Proposition 6.3.** *Let  $E$  be an elliptic curve over  $F$  with either good ordinary reduction or multiplicative reduction at every prime of  $F$  above  $p$ . Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with  $G = \text{Gal}(F_\infty/F)$ . Suppose that the following statements hold.*

- (i)  *$N$  is a closed normal subgroup of  $G$  which is contained in  $H$ , and there is a finite family of closed normal subgroups  $N_i$  ( $0 \leq i \leq r$ ) of  $G$  such that  $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$  and such that  $N_i/N_{i-1} \cong \mathbb{Z}_p$  for  $1 \leq i \leq r$ .*
- (ii)  *$G/N$  is a non-abelian pro- $p$   $p$ -adic Lie group without  $p$ -torsion. (In particular, the dimension of the  $p$ -adic Lie group  $G/N$  is necessarily  $\geq 2$ .)*
- (iii) *Set  $L_\infty := F_\infty^N$ . Suppose that  $X(E/L_\infty)$  is torsion over  $\mathbb{Z}_p[[G/N]]$ .*
- (iv) *Suppose that either (a) or (b) holds.*
  - (a)  *$E_{p^\infty}$  is not rational over  $F_\infty^{N_1}$  and  $X(E/L_\infty)$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module.*
  - (b) *For every  $v \in S$ , the decomposition group of  $\text{Gal}(L_\infty/F)$  at  $v$  has dimension  $\geq 2$ ,  $X(E/L_\infty)$  is a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module and  $r = 1$ .*

*Then  $X(E/F_\infty)$  is a faithful  $\mathbb{Z}_p[[G]]$ -module.*

*Proof.* We first note that by an iterative application of Lemma 6.1, it follows from condition (i) and (iii) that  $X(E/F_\infty)$  is torsion over  $\mathbb{Z}_p[[G]]$ . (Alternatively, one may apply an argument similar to that in [12, Theorem 2.3], noting that  $N$  is a solvable uniform pro- $p$  group.) Now suppose that condition (iv)(a) holds. By an induction on  $r$ , it suffices to prove the proposition in this case assuming  $r = 1$ . By another application of Lemma 6.1, one has that  $H_1(N, X(A/F_\infty)) = 0$ . This in turn implies that  $X(E/F_\infty)[\gamma_N - 1] = 0$ , where  $\gamma_N$  is a topological generator of  $N$ . As observed in the proof of Theorem 3.3,  $I_N$  is closed in  $\mathbb{Z}_p[[G]]$  and so  $\cap_{i \geq 1} I_{N_1}^i = 0$ . Therefore, by Lemma 2.2, we are reduced to proving that  $X(E/F_\infty)_N$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module. Now, applying an argument similar to that in [7, Lemma 2.4], we have that the dual of the cokernel of the map

$$\alpha : X(E/F_\infty)_N \longrightarrow X(E/L_\infty)$$

is contained in  $H^1(N, E(F_\infty)_{p^\infty})$ . By the first assumption in condition (iv)(a) and [34, Proposition 10], we have that  $H^0(N, E(F_\infty)_{p^\infty}) = E(L_\infty)_{p^\infty}$  is finite. On the other hand, it follows from Lemma 3.2 that

$$\text{corank}_{\mathbb{Z}_p} E(L_\infty)_{p^\infty} = \text{corank}_{\mathbb{Z}_p} H^1(N, E(F_\infty)_{p^\infty}) + \text{corank}_{\mathbb{Z}_p[[N]]} E(F_\infty)_{p^\infty}.$$

Therefore, it follows that  $H^1(N, E(F_\infty)_{p^\infty})$ , and hence the cokernel of  $\alpha$ , is finite. We may now combine Lemma 2.4 with the second assumption of condition (iv)(a) to conclude that  $X(E/F_\infty)_N$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module.

We now consider the case when condition (iv)(b) holds. As above, it suffices to show that  $X(E/F_\infty)_N$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module. Again, by the argument of [7, Lemma 2.4], one can show that the map

$$\alpha : X(E/F_\infty)_N \longrightarrow X(E/L_\infty)$$

has kernel which is cofinitely generated over  $\mathbb{Z}_p[[H/N]]$ , and cokernel which is cofinitely generated over  $\mathbb{Z}_p$ . In particular, by condition (ii) and Theorem 3.1, the dual of the cokernel of  $\alpha$  is psuedo-null over  $\mathbb{Z}_p[[G/N]]$ . Furthermore, in view of the first assumption of condition (iv)(b), one can apply a similar argument in the spirit of the proof of [30, Lemma 8.7] to show that the dual of the cokernel of  $\alpha$  is a finitely generated torsion  $\mathbb{Z}_p[[H/N]]$ -module, and therefore, is psuedo-null over  $\mathbb{Z}_p[[G/N]]$ . Hence we have

$$q(X(E/F_\infty)_N) = q(X(E/L_\infty)),$$

where  $q$  is the quotient functor from the category of finitely generated  $\mathbb{Z}_p[[G/N]]$ -modules to the category of finitely generated  $\mathbb{Z}_p[[G/N]]$ -modules modulo pseudo-null  $\mathbb{Z}_p[[G/N]]$ -modules. Since  $X(E/L_\infty)$  is completely faithful over  $\mathbb{Z}_p[[G/N]]$ , it follows that  $X(E/F_\infty)_N$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module, as required.  $\square$

**Remark 6.4.** It is clear from the proof that under condition (iv)(b), one actually shows that  $X(E/F_\infty)_N$  is a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module. However, due to the lack of an analogous result for completely faithful modules in the direction of Lemma 2.2, we are not able to deduce complete faithfulness of  $X(E/F_\infty)$  from the complete faithfulness of  $X(E/F_\infty)_N$ . This is also precisely the reason why we require the extra assumption that  $r = 1$  in condition (iv)(b).

In the next proposition, we mention the best we can do when we do not assume  $r = 1$  in condition (iv)(b) which might be of interest.

**Proposition 6.5.** *Retaining the assumptions (i), (ii) and (iii) of Proposition 6.3. Furthermore, we assume that the action of  $G$  on  $N_i/N_{i-1}$  by inner automorphism is given by a homomorphism  $\chi_i : G/N \longrightarrow \mathbb{Z}_p^\times$  for every  $i$ . Suppose that for every  $v \in S$ , the decomposition group of  $\text{Gal}(L_\infty/F)$  at  $v$  has dimension  $\geq 2$ , and suppose that  $X(E/L_\infty)$  is a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module. Then  $X(E/F_\infty)_N$  is a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module, and for  $i \geq 1$ ,  $H_i(N, X(E/F_\infty))$  is either a pseudo-null  $\mathbb{Z}_p[[G/N]]$ -module or a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module.*

*Proof.* The proof of Proposition 6.3 carries over to show that  $X(E/F_\infty)_N$  is a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module. By [22, Proposition 4.2],  $H_i(N, X(E/F_\infty))$  is a successive extension of twists of a

$\mathbb{Z}_p[[G/N]]$ -subquotient  $T$  of  $X(E/F_\infty)_N$  by a one dimensional character. Therefore, if  $H_i(N, X(E/F_\infty))$  is not a pseudo-null  $\mathbb{Z}_p[[G/N]]$ -module, then  $T$  cannot be a pseudo-null  $\mathbb{Z}_p[[G/N]]$ -module. Since  $X(E/F_\infty)_N$  is completely faithful over  $\mathbb{Z}_p[[G/N]]$ , so is  $T$ . It is not difficult to verify that every twist of  $T$  by a one dimensional character is also completely faithful over  $\mathbb{Z}_p[[G/N]]$ . Hence we may apply Lemma 2.5 to conclude that  $H_i(N, X(E/F_\infty))$  is completely faithful over  $\mathbb{Z}_p[[G/N]]$ .  $\square$

We also mention that it is clear from the proof of Proposition 6.3 that we can prove the following proposition for a general  $N$ .

**Proposition 6.6.** *Let  $E$  be an elliptic curve over  $F$  with either good ordinary reduction or multiplicative reduction at every prime of  $F$  above  $p$ . Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with  $G = \text{Gal}(F_\infty/F)$ . Suppose that the following statements hold.*

- (i)  $N$  is a closed normal subgroup of  $G$  which is contained in  $H$ .
- (ii)  $G/N$  is a non-abelian pro- $p$   $p$ -adic Lie group without  $p$ -torsion. (In particular, the dimension of the  $p$ -adic Lie group  $G/N$  is necessarily  $\geq 2$ .)
- (iii) Set  $L_\infty := F_\infty^N$ . Suppose that either (a) or (b) holds.
  - (a)  $E_{p^\infty}$  is not rational over  $F_\infty$  (note the slight difference here) and  $X(E/L_\infty)$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module.
  - (b) For every  $v \in S$ , the decomposition group of  $\text{Gal}(L_\infty/F)$  at  $v$  has dimension  $\geq 2$ , and  $X(E/L_\infty)$  is a completely faithful  $\mathbb{Z}_p[[G/N]]$ -module.

Then  $X(E/F_\infty)_N$  is a faithful  $\mathbb{Z}_p[[G/N]]$ -module.

The next control theorem is in the direction of a Hida deformation. We recall that  $A$  is the  $R$ -cofree Galois module attached to the Hida deformation as defined at the end of Section 4, where  $R = \mathbb{Z}_p[[X]]$ , and has the property that  $A[P] = E_{p^\infty}$  for some prime ideal  $P$  of  $R$ . As before, we denote by  $X(A/F_\infty)$  the dual Selmer group of the Hida deformation.

**Proposition 6.7.** *Let  $F_\infty$  be a strongly admissible  $p$ -adic Lie extension of  $F$  with Galois group  $G$ . Suppose that the following statements hold.*

- (i)  $G$  is non-abelian and has dimension  $\geq 2$ .
- (ii)  $X(E/F_\infty)$  is torsion over  $\mathbb{Z}_p[[G]]$ .
- (iii) Either (a) or (b) holds.
  - (a)  $E_{p^\infty}$  is not rational over  $F_\infty$  and  $X(E/F_\infty)$  is a faithful  $\mathbb{Z}_p[[G]]$ -module.
  - (b) For every  $v \in S$ , the decomposition group of  $G$  at  $v$  has dimension  $\geq 2$ , and  $X(E/F_\infty)$  is a completely faithful  $\mathbb{Z}_p[[G]]$ -module.

Then  $X(A/F_\infty)$  is faithful over  $R[[G]]$ .

*Proof.* The proof is essentially similar to that in Proposition 6.3. The only thing which perhaps requires additional attention is to show that the cokernel of the map

$$\beta : X(A/F_\infty)/P \longrightarrow X(E/F_\infty)$$

is finite under the assumption of condition (iii)(a). Note that the dual of its cokernel is contained in  $A(F_\infty)/P$ , where we write  $A(F_\infty) = A^{\text{Gal}(\bar{F}/F_\infty)}$ . Noting that  $A[P] = E_{p^\infty}$ , it then follows from (the dual of) Lemma 2.1 that

$$\text{corank}_{\mathbb{Z}_p} E(F_\infty)_{p^\infty} = \text{corank}_R A(F_\infty) + \text{corank}_{\mathbb{Z}_p} A(F_\infty)/P.$$

Again, by the first assumption of condition (iii)(a) and [34, Proposition 10], we have that  $E(F_\infty)_{p^\infty}$  is finite. Combining this observation with the above equation, we have that  $\text{corank}_{\mathbb{Z}_p} A(F_\infty)/P = 0$ , or equivalently, that  $A(F_\infty)/P$  is finite. This in turns implies that the cokernel of  $\beta$  is finite. The remainder of the proof proceeds as in Proposition 6.3.  $\square$

**Remark 6.8.** Of course, one can have a control theorem result for the faithfulness of the Selmer group of the Hida deformation for other specializations. In particular, one can also generalize the above control theorem to (appropriate) Selmer groups of more general deformations over  $\mathbb{Z}_p[[X_1, \dots, X_r]]$  and their various intermediate specializations as considered in [10].

We end the paper discussing an example to illustrate our control theorem results. Let  $p = 5$ . Let  $E$  be the elliptic curve 21a4 of Cremona's tables given by

$$y^2 + xy = x^3 + x$$

and let  $A$  be an elliptic curve 1950y1 of Cremona's tables

$$A : y^2 + xy = x^3 - 355303x - 89334583.$$

Let  $p = 5$ ,  $F = \mathbb{Q}(\mu_5)$  and  $L_\infty = F(A_{5^\infty})$ . As discussed in [2, Section 7], if  $X(E/F)$  is finite (as suggested by its  $p$ -adic  $L$ -function), then  $X(E/L_\infty)$  is a completely faithful  $\mathbb{Z}_5[[\text{Gal}(L_\infty/F)]]$ -module. *We will assume this latter property throughout our discussion here.* Let  $F_\infty$  be any strongly admissible 5-adic Lie extension of  $F$  which contains  $L_\infty$  and such that  $N = \text{Gal}(F_\infty/L_\infty)$  satisfies the condition in Proposition 6.3. It is not difficult to see that  $E_{5^\infty}$  is not rational over  $F_\infty$ . Hence we can apply Proposition 6.3 to conclude that  $X(E/F_\infty)$  is a faithful  $\mathbb{Z}_5[[\text{Gal}(F_\infty/F)]]$ -module. Examples of strongly admissible 5-adic extensions  $F_\infty$  that we may take are:

$$\begin{aligned} &\mathbb{Q}(A[5^\infty], 2^{5^\infty}), \quad \mathbb{Q}(A[5^\infty], 2^{5^\infty}, 3^{5^\infty}), \quad \mathbb{Q}(A[5^\infty], 3^{5^\infty}, 5^{5^\infty}), \quad \mathbb{Q}(A[5^\infty], 2^{5^\infty}, 3^{5^\infty}, 5^{5^\infty}) \\ &\mathbb{Q}(A[5^\infty], 2^{5^\infty}, 3^{5^\infty}, 5^{5^\infty}, 7^{5^\infty}), \quad \mathbb{Q}(A[5^\infty], 2^{5^\infty}, 3^{5^\infty}, 5^{5^\infty}, 7^{5^\infty}, 11^{5^\infty}), \quad \dots \text{ etc,} \\ &M_\infty(A[5^\infty], 2^{5^\infty}), \quad M_\infty(A[5^\infty], 2^{5^\infty}, 3^{5^\infty}), \quad M_\infty(A[5^\infty], 3^{5^\infty}, 5^{5^\infty}), \end{aligned}$$



$$M_\infty(A[5^\infty], 2^{5^{-\infty}}, 3^{5^{-\infty}}, 5^{5^{-\infty}}) \quad M_\infty(A[5^\infty], 2^{5^{-\infty}}, 3^{5^{-\infty}}, 5^{5^{-\infty}}, 7^{5^{-\infty}}), \dots \text{etc},$$

where  $M_\infty$  is any  $\mathbb{Z}_5^r$ -extension of  $F$  disjoint from  $F^{\text{cyc}}$  for  $1 \leq r \leq 2$ . However, at present, we are not able to determine whether or not  $X(E/F_\infty)$  is completely faithful for any of such  $F_\infty$ .

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